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Problem set 1 - Solutions

Exercise 1
Suppose the process for income is

$$y_t = \overline{y} + \varepsilon_t + \beta\varepsilon_{t-1} \tag{1}$$

Using the permanent income model studied in class, find the expression for $c_t - c_{t-1}$ as function of $\varepsilon_{t.}$

Answer
From class, we know that

$$c_t - c_{t-1} = c_t - E_{t-1}c_t \tag{2}$$

$$= \frac{r}{1+r}\sum_{s=0}^{\infty}\frac{E_t y_{t+i} - E_{t-1}y_{t+i}}{(1+r)^i} \tag{3}$$

From the process for y_t we know that

$$E_t y_t - E_{t-1}y_t = \varepsilon_t \tag{4}$$

$$E_t y_{t+1} - E_{t-1}y_{t+1} = \beta\varepsilon_t \tag{5}$$

$$E_t y_{t+i} - E_{t-1}y_{t+i} = 0 \qquad i \geq 2 \tag{6}$$

Hence

$$c_t - c_{t-1} = \frac{r}{1+r}\varepsilon_t(1 + \frac{\beta}{1+r}) \tag{7}$$

Exercise 2
Use the permanent income model studied in class, and the stochastic process $y_{t+1} = \lambda y_t + \varepsilon_{t+1}$ that we also used in class. Show that the consumption function (linking c_t to its determinants y_t and A_t) is

$$c_t = rA_t + \frac{r}{1+r-\lambda}y_t \tag{8}$$

What happens if $\lambda = 1$ or $\lambda = 0$?

Answer:

$$c_t = rA_t + r\frac{1}{1+r}\sum_{i=0}^{\infty}\frac{y_{t+i}}{(1+r)^i} \tag{9}$$

$$= rA_t + r\frac{1}{1+r}\sum_{i=0}^{\infty}\frac{\lambda^i y_t}{(1+r)^i} \tag{10}$$

$$= rA_t + r\frac{1}{1+r}y_t\frac{1}{1-\frac{\lambda}{1+r}} \tag{11}$$

$$= rA_t + \frac{r}{1+r-\lambda}y_t \tag{12}$$

For the effects of $\lambda = 0$ and $\lambda = 1$, see Exercise 1.

Exercise 3

Consider the two-period model

$$\max_{c_1,c_2} U = u(c_1) + \frac{1}{1+\rho}u(c_2) \tag{13}$$

s.t.

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{14}$$

The utility function is

$$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \quad \text{for } \sigma \geq 0, \quad \sigma \neq 1 \tag{15}$$

$$u(c) = \log c \quad \text{for } \sigma = 1 \tag{16}$$

a) Write the Lagrangean of the problem and show that λ (the Lagrange multplier on the intertemporal budget constraint) is equal to the marginal utility of consumption in period 1.

Answer:

The Lagrangean is:

$$\mathcal{L} = \frac{c_1^{1-\sigma}-1}{1-\sigma} + \frac{1}{1+\rho}\frac{c_2^{1-\sigma}-1}{1-\sigma} + \lambda\left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r}\right] \tag{17}$$

Differentiating with respect to c_1 and c_2 we get the first order conditions

$$c_1^{-\sigma} = \lambda \tag{18}$$

$$c_2^{-\sigma} = \lambda \left(\frac{1+\rho}{1+r}\right) \tag{19}$$

b) Show that the Euler equation is

$$\left(\frac{c_1}{c_2}\right)^{\sigma} = \frac{1+\rho}{1+r} \tag{20}$$

Answer:
Combining (18) and (19) we get the result.
c) Show that the elasticity of intertempral substitution

$$-\frac{\partial \log(c_1/c_2)}{\partial \log(1+r)} \tag{21}$$

is equal to $1/\sigma$. Interpret the intertemporal elasticity of substitution.
Answer:
Take the logs of both sides of (20) and then differentiate. To interpret the intertemporal elasticity of substitution, note that $1+r$ can be interpreted as the price of consumption in 1 relative to consumption in 2: the higher the interest rate, the more of c_2 I give up if I consume one extra unit of c_1. hence the intertemporal elasticity of substitution gives the percentage change in (c_1/c_2) in response to an increase in the interest rate by one percentage point (recall that $d\log(1+r) \approx dr$).
d) Now find the consumption function for c_1, relating the latter with y_1 and y_2
Answer:
Just use the intertemporal budget constraint (14) to replace c_2 with $y_2 + (1+r)(y_1 - c_1)$, to get

$$c_1 = \left(y_1 + \frac{y_2}{1+r}\right)(1+\rho)^{\frac{1}{\sigma}}\left[(1+r)^{\frac{1-\sigma}{\sigma}} + (1+\rho)^{\frac{1}{\sigma}}\right]^{-1} \tag{22}$$

e) Show the effect on savings in period 1 of a change in the interest rate. Assume first that $y_2 = 0$, then show how the conclusions are influenced when y_2 is positive.
Answer:
If $y_2 = 0$, from the equation above we have

$$c_1 = y_1(1+\rho)^{\frac{1}{\sigma}}\left[(1+r)^{\frac{1-\sigma}{\sigma}} + (1+\rho)^{\frac{1}{\sigma}}\right]^{-1} \tag{23}$$

and

$$s = y_1 - c_1 \tag{24}$$

An increase in r increases c_1 and reduces savings if $\sigma > 1$, otherwise if $\sigma < 1$. The intuition is the following. When $y_2 = 0$, a change in r has two effects on c_1. The first is a **substitution effect**: when r increases, the relative price of c_1 (relative to c_2) increases; this indiuces the individual to reduce c_1 and increases savings. The second is an **income effect**: when r increases, the individual can get the same c_2 with lower savings; this pushes c_1 up and savings down. The substitution effect is stronger, the higher the intertemporal elasticity of substitution, i.e. the lower σ is. Hence, if $\sigma < 1$, the substitution effect prevails, and savings increases.

Now assume that $y_2 > 0$: there is now a third effect, the **wealth effect:** an increase in r reduces wealth, given y_2: this induces the consumer to reduce c_1, hence to increase savings. The wealth effect then reinforces the substitution effect.

Exercise 4

Consider a consumer who lives for two periods, with incomes y_1 and y_2. The consumer can borrow at rate r_D, while she can save at the rate r_A, with $r_D > r_A$.

a) Show the budget constraint of the consumer in the space (c_1, c_2).

Answer:

See Figure

b) Determine the necessary and sufficient condition for the optimal consumption to be precisely (y_1, y_2). These conditions are in the form of inequalities of the derivatives of $U(c_1, c_2)$ evaluated at (y_1, y_2).

Answer:

The condition is

$$r_A \leq -\frac{U_{C_1}(C_1, C_2)}{U_{C_2}(C_1, C_2)} \leq r_D \tag{25}$$

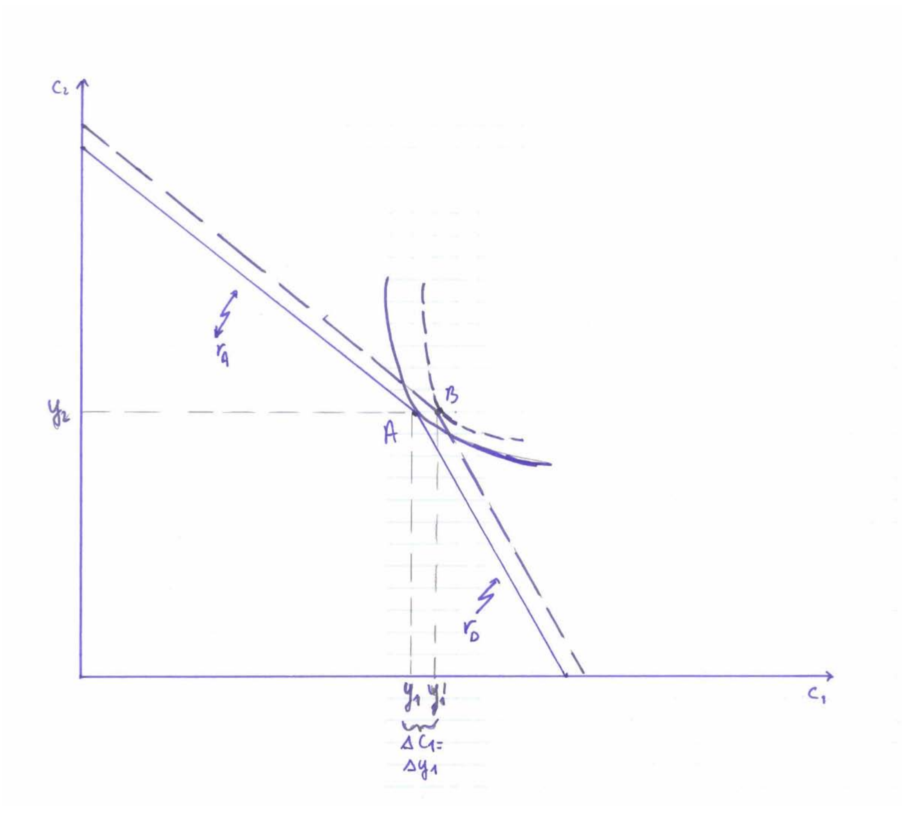
c) Suppose that the inequalities in b) are satisfied. Show graphically that if y_1 increases by a small amount Δy_1, then $\Delta c_1/\Delta y_1 = 1$ and $\Delta c_2/\Delta y_1 = 0$, which is closer to the keynesian consumption function. Explain.

Answer:

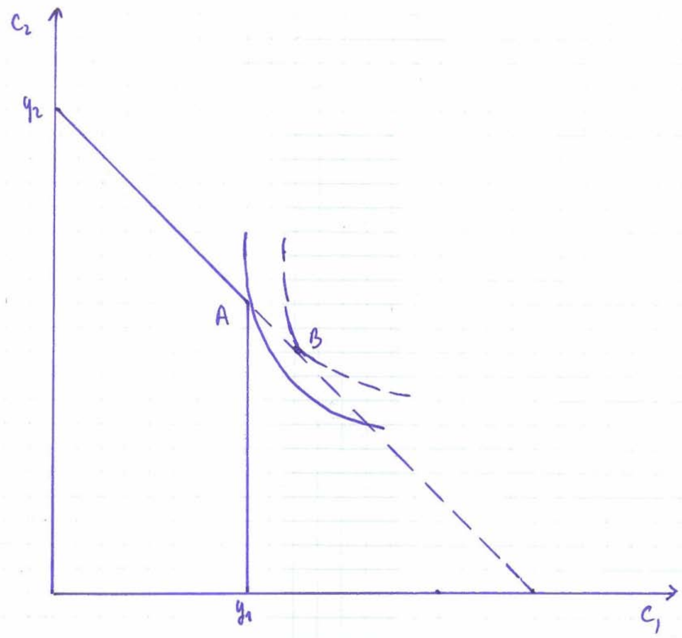
See Figure . The consumer wanted to move closer to consumption smoothing but, if the inequality above was satisfied, it was too costly. The rise in y_1 allows the consumer to get closer to consumption smoothing.

d) Now suppose that the consumer cannot borrow at all, i.e. that $r_D = \infty$. Explain with a graph.

Answer:



c_2
r_A
y_2
A
B
r_D
y_1 y_1'
c_1
$\Delta c_1 = \Delta y_1$



c_2
y_2
A
B
y_1
c_1

See Figure .

Exercise 5

Consider an individual who lives three periods ("young", "middle-aged" and "old"). Her incomes in the three periods are $Y_1 = Y$, $Y_2 = (1+\gamma)Y$, and $Y_3 = 0$. The individual wants to do perfect consumption smoothing, i.e. $C_1 = C_2 = C_3$. The interest rate is 0.

a) Compute savings in each periods.

Answer:

$$\overline{C} = C_1 = C_2 = C_3 = Y\left(\frac{2+\gamma}{3}\right) \tag{26}$$

Hence

$$S_1 = Y - \overline{C} = Y\left(\frac{1-\gamma}{3}\right) \tag{27}$$

$$S_2 = Y(1+\gamma) - \overline{C} = Y\left(\frac{1+2\gamma}{3}\right) \tag{28}$$

$$S_3 = -\overline{C} = -Y\left(\frac{2+\gamma}{3}\right) \tag{29}$$

b) Suppose that there is no population growth, i.e. at each point in time one third of the population alive belongs to the three ages, i.e. one third earns Y, one third $Y(1+\gamma)$, and one third 0. What is aggregate savings in the economy?

Answer:

Let $\widetilde{S}$ indicate aggregate savings.

$$\widetilde{S} = S_1 + S_2 + S_3 = 0 \tag{30}$$

c) Suppose that a pension system is introduced whereby each young and middle-aged is forced to save A, and gets $2A$ when old. What happens to the savings of the individuals?

Answer:

It depends. If $A < S_1$, the individual will continue to save S_1; if instead $A > S_1$, her savings will increase. The same happens for the middle-aged.

d) Now suppose that population grows at rate n each period. Compute aggregate savings of the economy. Show how aggregate savings change when γ increases. Explain, and compare with your answer to b).

Answer:

Let $\widetilde{S}_1$ indicate aggregate savings by the young, and similarly for $\widetilde{S}_2$ and $\widetilde{S}_3$. At each point in time, if we normalize the population of the old at 1, there are $1+n$ middle-aged and $(1+n)^2$ yopung. Hence

$$\begin{aligned} \widetilde{S} &= \widetilde{S}_1 + \widetilde{S}_2 + \widetilde{S}_3 && (31) \\ &= (1+n)^2 S_1 + (1+n) S_2 + S_3 && (32) \\ &= Y \left[\frac{n^2(1-\gamma) + 3n}{3} \right] > 0 && (33) \end{aligned}$$

When γ increases, aggergate savings falls. The reason is that the wealth of the young increases (they anticipate that their income when middle aged wil increase), hence their consumption increases; given their income, their savings when young falls. A similar reasoning applies to the middle aged. Because the two groups that save aremore numerous, aggergate savings increases. .

e) What is the rate of growth of aggregate income in this economy?

Answer:

$$Y_t = (1+n)^2 Y + (1+n)(1+\gamma) Y \quad (34)$$

$$Y_{t+1} = (1+n)^3 Y + (1+n)^2 (1+\gamma) Y \quad (35)$$

Hence

$$\begin{aligned} \widehat{Y}_{t+1} &\equiv \frac{Y_{t+1} - Y_t}{Y_t} && (36) \\ &= 1+n && (37) \end{aligned}$$

f) Does your answer support the often-heard statement that "in order to grow more a country has to save more"?

Answer:

This statement would be correct outside the steady-state in a Solow growth model. Here, aggregate savings is 0 if there is no population growth. If there is population growth, both growth and aggregate savings are positive, but both are a consequence of population grotwh: it is not the case that higher savings *causes* higher growth.